Commutative Rational Term Rewriting

Mamoru Ishizuka[1], Takahito Aoto[1][0000−0003−0027−0759], and Munehiro Iwami[2][0000−0001−9925−450X]

[1] Niigata University, Niigata, Japan
{ ishizuka@nue., aoto@ }ie.niigata-u.ac.jp
[2] Simane University, Shimane, Japan
munehiro@cis.shimane-u.ac.jp

Abstract. Term rewriting for rational terms, i.e. infinite terms with a finite number of different subterms, has been considered e.g. in Corradini & Gadducci (1998) and Aoto & Ketema (2012). In this paper, we consider rational term rewriting by a set of commutativity rules i.e. rules of the form $f(x, y) \to f(y, x)$, based on the framework of Aoto & Ketema (2012). A rewrite step with a commutativity rule is specified via a regular set of redex positions, thus via a finite automaton. We present some finite automata constructions that correspond to (in particular) taking inverse rewrite steps, merging two branching rewrite steps, and merging two consecutive rewrite steps. As a corollary, we show that rational rewrite steps by the commutativity rules are closed under taking equivalence of the rewrite steps.

Keywords: Rational term rewriting · Commutativity · Finite automata.

1 Introduction

Term rewriting systems (*TRSs*) is a computational model based on equational logic [3]. Besides the standard rewriting formalism, many variations and extensions have been considered. One direction of such extensions is towards incorporating infinitary phenomena for various aspects of computation. In particular, there is a long history of investigations on infinitary rewriting where (infinitary long) rewriting of infinite terms is considered, and that on graph rewriting where rewriting of (cyclic or acyclic) term graphs is considered (see e.g. [11, 2, 10]). In this paper, we consider yet another such a formalism of rewriting dealing with infinitary phenomena, *rewriting of rational terms*.

Rational terms are infinite terms with a finite number of different subterms [6, 8, 5, 1]. Unraveling a cyclic term graph into an infinite term yields a term that is rational, and rational terms are represented finitely [6–8]. In [1], a framework of rational term rewriting has been considered, and some basic decidability results concerning computations of the rewriting are given. In this framework, a rewrite step is specified by a rewrite rule and regular set of redex positions; the reduct is obtained by simultaneously rewriting at the redex positions. In this paper, we consider (a variant of) rational rewriting by *commutativity rules*—rewrite rules of the form $f(x, y) \to f(y, x)$. Commutative rewriting is a basis of the C-unification and AC-unification which have been well-studied in the case of the standard rewriting [4]. To the best of our knowledge, however, commutative rewriting has been yet beyond the scope of the study in rational term rewriting.

We present some finite automata constructions that correspond to (in particular) taking inverse rewrite steps, merging two branching rewrite steps, and merging two consecutive rewrite steps of rational term rewriting by the commutativity rules. It seems such constructions have not been studied in literature. As a corollary, we show that rational rewrite steps by the commutativity rules are closed under taking equivalence of the rewrite steps.

2 Preliminaries

In this section, we explain notions and notations that will be used in this paper. Our definitions and notation follow [1].

2.1 Finite Automata

Let Σ be a finite set of symbols. An *empty* sequence is denoted by ε and the *concatenation* of finite sequence $p, q \in \Sigma^*$ is denoted by $p.q$. A *deterministic finite automaton* (*DFA* for short) is a tuple $M = \langle Q, \Sigma, \delta, q_0, F\rangle$ where Q is a set of states, Σ is a set of input symbols, $\delta : Q \times \Sigma \to Q$ is a transition function, $q_0 \in Q$ is an initial state and $F \subseteq Q$ is a set of final states. The homomorphic extension of δ is denoted by $\hat{\delta} : Q \times \Sigma^* \to Q$. Let $\mathcal{L}(M, q_i) \subseteq \Sigma^*$ ($q_i \in Q$) be the smallest set such that (i) $\varepsilon \in \mathcal{L}(\mathcal{M}, q_0)$, and (ii) if $p \in \mathcal{L}(\mathcal{M}, q)$ and $\delta(q, a) = q_i$ then $p.a \in \mathcal{L}(\mathcal{M}, q_i)$. The *language* of a DFA M is given by $\mathcal{L}(M) = \bigcup_{q_i \in F} \mathcal{L}(M, q_i)$. Let $M_1 = \langle Q_1, \Sigma, \delta_1, q_1, F_1\rangle, M_2 = \langle Q_2, \Sigma, \delta_2, q_2, F_2\rangle$ be DFAs and suppose $\approx \subseteq Q_1 \times Q_2$. The relation $\approx$ is a *bisimulation relation* if (i) $q_1 \approx q_2$, (ii) $p \approx q$ implies $\delta_1(p, a) \approx \delta_2(q, a)$ for any $a \in \Sigma$ and (iii) if $p \approx q$, then $p \in F_1$ iff $q \in F_2$. Two DFAs M_1, M_2 are *bisimilar* ($M_1 \approx M_2$) if there exists a bisimulation relation. The following property is known (e.g. [9]).

Proposition 1. *Let M_1, M_2 be DFAs. Then, $M_1 \approx M_2$ iff $\mathcal{L}(M_1) = \mathcal{L}(M_2)$.*

2.2 Rational Terms

We denote a set of arity-fixed *function symbols* by $\mathcal{F}$ and a countably infinite set of *variables* by $\mathcal{V}$, where $\mathcal{F} \cap \mathcal{V} = \emptyset$. The *arity* of function symbol $f \in \mathcal{F}$ is denoted by $\mathrm{arity}(f)$. Let $\mathcal{F}_n = \{f \in \mathcal{F} \mid \mathrm{arity}(f) = n\}$. Function symbols in $\mathcal{F}_0$ are called *constants. We assume there exists some $n \geq 0$ such that* $\mathrm{arity}(f) \leq n$ *for all $f \in \mathcal{F}$.* We denote the set of positive integers by $\mathbb{N}_+$, and the set of finite sequence of positive integers by $\mathbb{N}_+^*$. An *infinite term* t over $\mathcal{F}$ and $\mathcal{V}$ is a partial function from $\mathbb{N}_+^*$ to $\mathcal{F} \cup \mathcal{V}$ such that (i) $t(\varepsilon)$ is defined, and (ii) $t(p.i)$ ($i \in \mathbb{N}$) is defined iff $t(p) \in \mathcal{F}_n$ and $1 \leq i \leq n$ for some n. The set of infinite terms is denoted by $\mathcal{T}_{inf}(\mathcal{F}, \mathcal{V})$. Infinite terms are often abbreviated as terms below. The set $\mathrm{Pos}(t)$ of *positions* of a term t is the domain of the partial function t. In particular, ε is called the *root* position. A term t is a *finite* term if $\mathrm{Pos}(t)$ is a finite set. The symbol $t(p) \in \mathcal{F} \cup \mathcal{V}$ is called *the symbol at the position* p. $\mathcal{V}(t)$ is the set of variables appearing in t, that is $\mathcal{V}(t) = \{t(p) \in \mathcal{V} \mid p \in \mathrm{Pos}(t)\}$. A *subterm* $t|_p$ of t at the position $p \in \mathrm{Pos}(t)$ is a mapping given by $t|_p(q) = t(p.q)$. A term $t \in \mathcal{T}_{inf}(\mathcal{F}, \mathcal{V})$ is *rational* if the set of subterms $\{t|_p \mid p \in \mathrm{Pos}(t)\}$ of t is finite. Clearly, finite terms are always rational.

Example 1. Let $\mathsf{g}, \mathsf{h} \in \mathcal{F}_1$. Let s be a partial mapping $\{\varepsilon \mapsto \mathsf{g}, 1 \mapsto \mathsf{h}, 1.1 \mapsto x\}$. It is easy to see that s is a term; furthermore, the domain of s is a finite set $\{\varepsilon, 1, 1.1\}$, and thus s is a finite term. In usual notation, $s = \mathsf{g}(\mathsf{h}(x))$. Let t be a partial mapping given by $t(1^n) = \mathsf{g}$ for any $n \geq 0$, and undefined otherwise. Here, 1^n is the sequence of 1's of length n. Intuitively, t is an infinite term $t = \mathsf{g}(\mathsf{g}(\mathsf{g}(\cdots)))$. In fact, the set of subterm of t is given by $\{t\}$ (i.e. all subterms are equal to t), thus t is a rational term. Similarly, if we take $u = \mathsf{g}(\mathsf{h}(\mathsf{g}(\mathsf{h}(\cdots))))$, then the set of subterms of u equals to $\{u, \mathsf{h}(u)\}$, and hence u is a rational term. Clearly, $u|_{1^{2n}} = u$ and $u|_{1^{2n+1}} = \mathsf{h}(u)$ hold for each $n \geq 0$. Now, let $\mathsf{f} \in \mathcal{F}_2$ in addition, and let v be a mapping $v = \{1^i \mapsto \mathsf{f} \mid i \geq 0\} \cup \{1^i.2.1^j \mapsto \mathsf{g} \mid i \geq 0, j < i\} \cup \{1^i.2.1^i \mapsto x \mid i \geq 0\}$. Then v is an infinite term that is not rational. □

A *substitution* is a mapping $\sigma : \mathcal{V} \to \mathcal{T}_{inf}(\mathcal{F}, \mathcal{V})$ such that its domain $\mathrm{dom}(\sigma) = \{x \mid \sigma(x) \neq x\}$ is finite. A substitution is identified with its homomorphic extension; as usual, $\sigma(t)$ is rewritten as $t\sigma$.

A *regular system* is a finite set $E = \{x_1 = t_1, \ldots, x_n = t_n\}$ of equations such that the left hand sides $x_1, \ldots, x_n$ are mutually distinct variables and t_i is a finite term for all $1 \leq i \leq n$. We set its domain as $\mathcal{D}om(E) = \{x_1, \ldots, x_n\}$ and its range as $\mathcal{R}an(E) = \{t_1, \ldots, t_n\}$. We write $E(y) = t$ if $y = t \in E$. A variable $x_i \in \mathcal{D}om(E)$ is *looping* if the exists $1 \leq i_1, \ldots, i_k \leq n$ such that $x_i = t_{i_1}$, and for each $1 \leq j \leq k$, $t_{i_j} = x_{i_{(j \bmod k)+1}}$ holds. Otherwise, $x_i \in \mathcal{D}om(E)$ is *non-looping*. Let $\perp$ be a new constant and $\mathcal{F}_\perp = \mathcal{F} \cup \{\perp\}$. We define a term $E^\star(x_i) \in T_{inf}(\mathcal{F}_\perp, \mathcal{V})$ for each $x_i \in \mathcal{D}om(E)$ as follows:

$$E^\star(x_i)(p) = \begin{cases} t_i(p) & \text{if } p \in \mathrm{Pos}(t_i) \text{ and } t_i(p) \notin \mathcal{D}om(E) \\ \perp & \text{if } t_i(p) = x_j \in \mathcal{D}om(E) \text{ and } x_j \text{ is looping} \\ E^\star(x_j)(q) & \text{if there exists } p' \text{ such that } p = p'.q \\ & \text{and } t_i(p') = x_j \in \mathcal{D}om(E) \text{ and } x_j \text{ is non-looping} \\ \text{undefined} & \text{otherwise} \end{cases}$$

If $E^\star(x) = t$ then the pair $\langle E, x \rangle$ (E_x in short) is called a *representation* of t.

Example 2. Let s, u be terms in Example 1. $\{y = \mathsf{g}(z), z = \mathsf{h}(x)\}_y$ and $\{y = \mathsf{g}(\mathsf{h}(x))\}_y$ are representations of s. Let $E = \{x = \mathsf{g}(y), y = \mathsf{h}(x)\}$. Then $u = E^\star(x)$ and E_x is a representation of u. If we identify a mapping E with its homomorphic extension, then we have $E^0(x) = x$, $E^1(x) = \mathsf{g}(y)$, $E^2(x) = E(E(x)) = E(\mathsf{g}(y)) = \mathsf{g}(E(y)) = \mathsf{g}(\mathsf{h}(x))$, $E^3(x) = \mathsf{g}(\mathsf{h}(\mathsf{g}(y)))$, ... whose limit will be u. On the other hand, if we set $F = \{x = y, y = x\}$, then $F^0(x), F^1(x), F^2(x), F^3(x), \ldots$ are $x, y, x, y, \ldots$, which does not converge. Note we obtain $F^\star(x) = \bot$. Note that a non-looping regular system can be obtained by replacing every equation $x = t$ with x looping by $x = \bot$. □

Henceforth, we assume $\mathcal{F}$ contains the constant $\bot$.

The following proposition on regular systems will be used later.

Proposition 2 (Lemma 3.3 of [1]). *Let E and F be regular systems and suppose there exists a surjection $\delta : \mathcal{D}om(E) \to \mathcal{D}om(F)$ such that $\delta(y) = \delta(s) \in F$ for every $y = s \in E$, where δ is homomorphically extended to a substitution on terms in the usual way. Then, $E^\star(y) = F^\star(\delta(y))$ for every $y \in \mathcal{D}om(E)$.*

Let E be a regular system and $x \in \mathcal{D}om(E)$. Then, define $\mathcal{U}_E(x)$ as the smallest set satisfying: (1) $x \in \mathcal{U}_E(x)$, and (2) if $y \in \mathcal{U}_E(x)$ and $y = t \in E$ then $\mathcal{V}(t) \cap \mathcal{D}om(E) \subseteq \mathcal{U}_E(x)$. We write $y \sqsubseteq_E x$ if $y \in \mathcal{U}_E(x)$. If E is obvious from the context, the subscript $_E$ may be omitted. Next, for each $y \sqsubseteq x$ we define $SP_{E_x}(y)$ as the smallest set satisfying: (1) $\varepsilon \in SP_{E_x}(x)$ and (2) if $p \in SP_{E_x}(z)$ and there exists $z = t \in E$ such that $t|_q = y$, then $p.q \in SP_{E_x}(y)$. We also define $SP_{E_x}(y) = \emptyset$ for $y \not\sqsubseteq x$. Intuitively, $SP_{E_x}(y)$ denotes the set of positions in $E^\star(x)$ corresponding to $y \in \mathcal{D}om(E)$. Finally, we put for any set $W \subseteq \mathcal{U}_E(x)$, $SP_{E_x}(W) = \bigcup_{y \in W} SP_{E_x}(y)$; note $SP_{E_x}(U \cup W) = SP_{E_x}(U) \cup SP_{E_x}(W)$ and $SP_{E_x}(U \setminus W) = SP_{E_x}(U) \setminus SP_{E_x}(W)$ follow from the definition.

Example 3. Let $E = \{x = \mathsf{f}(y, x), y = \mathsf{g}(z), z = \mathsf{h}(y)\}$ be a regular system. Then $\mathcal{U}_E(x) = \{x, y, z\}, \mathcal{U}_E(y) = \mathcal{U}_E(z) = \{y, z\}$. If we put $E^\star(y) = \mathsf{g}(\mathsf{h}(\mathsf{g}(\mathsf{h}(\cdots)))) = s$, then $E^\star(x) = \mathsf{f}(s, \mathsf{f}(s, \mathsf{f}(\cdots)))$. Now, $SP_{E_x}(x) = \{2^n \mid n \geq 0\}$, $SP_{E_x}(y) = \{2^n.1^{2m+1} \mid n, m \geq 0\}$ and $SP_{E_x}(z) = \{2^n.1^{2m+2} \mid n, m \geq 0\}$. □

A regular system $E = \{x_1 = t_1, \ldots, x_n = t_n\}$ is *canonical* if E satisfies the condition: for each $1 \leq i \leq n$, either (i) $t_i \in \mathcal{V} \setminus \mathcal{D}om(E)$, or (ii) $t_i = f(y_1, \ldots, y_m)$ for some $f \in \mathcal{F}_m$ and $y_1, \ldots, y_m \in \mathcal{D}om(E)$. We say a representation $\langle E, x \rangle$ (or E_x) is canonical if so is E. It is known that from any regular system E one can construct a canonical regular system F such that (i) $\mathcal{D}om(E) \subseteq \mathcal{D}om(F)$, (ii) $E^\star(x) = F^\star(x)$ for all $x \in \mathcal{D}om(E)$, and (iii) $SP_{E_x}(y) = SP_{F_x}(y)$ for all $x, y \in \mathcal{D}om(E)$ such that $y \sqsubseteq x$.

2.3 Rational Term Rewriting

A pair $\langle l, r \rangle$, written also as $l \to r$, of finite terms l and r is a *rewrite rule* if $l \notin \mathcal{V}$ and $\mathcal{V}(l) \supseteq \mathcal{V}(r)$. A *term rewriting system* (*TRS* for short) is a finite set of rewrite rules. A TRS $\mathcal{R}$ is said to be *orthogonal* if l is linear term (any variable occurs at most once) for any $l \to r \in \mathcal{R}$, and there is no overlaps between rules, i.e. $l|_p$ and l' does not unify (w.l.o.g. assuming variables are disjoint) for rewrite rules $l \to r, l' \to r' \in \mathcal{R}$ and for each non-variable position p in l (when $l \to r = l' \to r'$, we moreover assume $p \neq \varepsilon$).

Definition 1. *Let $\mathcal{R}$ be an orthogonal TRS and s, t be rational terms. We have a* development *rewrite step $s \multimap_{\mathcal{R}} t$ if there exist representations E_x and F_x of* *s and t, resp., such that* $\mathcal{D}om(E) = \mathcal{D}om(F)$*, and a set* $W \subseteq \mathcal{D}om(E)$ *such that (1)* $E(y) = F(y)$ *for any* $y \in \mathcal{D}om(E) \setminus W$ *and (2) for any* $y \in W$*, there exist a rewrite rule* $l \to r \in \mathcal{R}$ *and a substitution* ρ *such that* $E(y) = l\rho$ *and* $F(y) = r\rho$*.*

We say that the rewrite step $s \multimap\!\!\to t$ is specified by $\langle E_x, F_x, W\rangle$, or $s \multimap\!\!\to t$ is a rewrite step obtained by applying the rewrite rules on W of E_x. If $\mathcal{R}$ is clear from the context, $s \multimap\!\!\to_{\mathcal{R}} t$ is abbreviated as $s \multimap\!\!\to t$. The set of *redex positions* of the rewrite step is given by $\Delta = SP_{E_x}(W)$, and we write $s \multimap\!\!\to^{\Delta} t$ to make the redex positions explicit. Note that a rewrite step may be specified by multiple representations.

Example 4. Let $\mathcal{F} = \{\mathsf{f}, \mathsf{g}, \mathsf{h}, \bot\}$ and $\mathcal{R} = \{\mathsf{f}(x,y) \to \mathsf{f}(y,x), \mathsf{g}(x,y) \to \mathsf{g}(y,x)\}$. Let $E = \{x = \mathsf{f}(x,y), y = \mathsf{g}(y,y)\}, F = \{x = \mathsf{f}(y,x), y = \mathsf{g}(y,y)\}$ be regular systems. Let $W = \{x\}$. Then, we have a rewrite step $s \multimap\!\!\to^{\Delta} t$, where $s = E^{\star}(x), t = F^{\star}(x)$ and $\Delta = SP_{E_x}(W) = \{1^n \mid n \geq 0\}$. This rewrite step is specified by $\langle E_x, F_x, W\rangle$. Let $E' = \{x = \mathsf{f}(z,y), z = \mathsf{f}(x,y), y = \mathsf{g}(y,y)\}, F' = \{x = \mathsf{f}(z,y), z = \mathsf{f}(y,x), y = \mathsf{g}(y,y)\}$ be regular systems. Then we have $E'^{\star}(x) = s$. Thus, by applying the rewrite rule on $W' = \{z\}$ of E'_x, we obtain a rewrite step $s \multimap\!\!\to^{\Gamma} u$, where $u = F'^{\star}(x)$ and $\Gamma = SP_{E'_x}(W') = \{1^{2n+1} \mid n \geq 0\}$. Lastly, suppose $G = \{x = \mathsf{f}(z,y), y = \mathsf{g}(z,x), z = \mathsf{h}(z)\}$ and $H = \{x = \mathsf{f}(y,z), y = \mathsf{g}(x,z), z = \mathsf{h}(z)\}$. Then, we have $G^{\star}(x) \multimap\!\!\to H^{\star}(y)$. The step $G^{\star}(x) \multimap\!\!\to H^{\star}(y)$ is specified by $\langle E_x, F_x, \{x,y\}\rangle$. As in the last example, different rewrite rules can be employed in a single development rewrite step. □

Remark 1. In [1], a (standard) rewrite step $s \to t$ is defined in such a way that a single rewrite rule is allowed to use in a rewrite step; the restriction is needed to deal with rewriting of possibly non-orthogonal TRS in general (see Remarks 4.3 and 4.4 in [1]). Contrast to this, in the development rewrite step $s \multimap\!\!\to t$, different rewrite rules $l \to r \in \mathcal{R}$ can be employed depending on $y \in W$. Note, however, because of the orthogonality, there can not be multiple candidates for such a rewrite rule for each $y \in W$.

In this paper, we focus on development rewrite steps by a set of commutativity rewrite rules, i.e. rules of the form $f(x,y) \to f(y,x)$. It should be also clear that any development rewrite step can be specified on canonical representations because of the form of the commutativity rules. Thus, *we will w.l.o.g. specify a rewrite step via canonical representations.*

2.4 Products of Canonical Regular Systems

In this subsection, we present some basic properties of the product construction of canonical regular systems, which will be used in the subsequent proofs.

Definition 2 (product of canonical regular systems). *Let* E, F *be canonical regular systems. We define the product* $E \times F$ *of* E *and* F *as follows.* $E \times F = \{\langle x,y\rangle = f(\langle x_1,y_1\rangle, \ldots, \langle x_n,y_n\rangle) \mid x = f(x_1,\ldots,x_n) \in E$ *and* $y = f(y_1,\ldots,y_n) \in F\} \cup \{\langle x,y\rangle = z \mid x = z \in E, y = z \in F$ *and* $z \notin \mathcal{D}om(E) \cup \mathcal{D}om(F)\}$*. Now, by regarding the pairs of variables as variables, we treat* $E \times F$ *as a canonical regular system.*

The following lemmas characterizes the term represented by $(E \times F)_{\langle x,y\rangle}$ and the positions in it, in terms of those in E_x.

Lemma 1. *Let E, F be canonical regular systems and $x \in \mathcal{D}om(E)$, $y \in \mathcal{D}om(F)$ such that $E^\star(x) = F^\star(y)$. Then, $E^\star(x) = (E \times F)^\star(\langle x, y\rangle)$.*

Lemma 2. *Let E, F be canonical regular systems and $x \in \mathcal{D}om(E)$, $y \in \mathcal{D}om(F)$ such that $E^\star(x) = F^\star(y)$. Let $W \subseteq \mathcal{D}om(E)$. Then, $SP_{E_x}(W) = SP_{(E\times F)_{\langle x,y\rangle}}(W \times \mathcal{D}om(F))$.*

Using these lemmas, we can characterize rewrite steps of the products.

Lemma 3. *Let $\mathcal{R}$ be a TRS and E_x a canonical representation of s. Suppose a rewrite step $s \mathrel{\circ\!\!\rightarrow}^{\Gamma}_{\mathcal{R}} t$ is obtained by applying the rewrite rules on $W \subseteq \mathcal{D}om(E)$ of E_x. Let F be a canonical regular system such that $s = F^\star(y)$. Then, $(E \times F)_{\langle x,y\rangle}$ is a representation of s, and the rewrite step $s \mathrel{\circ\!\!\rightarrow}^{\Gamma}_{\mathcal{R}} t$ is obtained by applying the rewrite rules on $W \times \mathcal{D}om(F)$ of $(E \times F)_{\langle x,y\rangle}$.*

Proof. By the assumption, $s = E^\star(x)$ and $\Gamma = SP_{E_x}(W)$. Then by Lemma 1, we have $s = (E \times F)^\star(\langle x, y\rangle)$. Moreover, by Lemma 2, $\Gamma = SP_{E_x}(W) = SP_{(E\times F)_{\langle x,y\rangle}}(W \times \mathcal{D}om(F))$. Thus the claim follows. $\square$

3 Automata for Inverse Rewrite Steps

In what follows, we consider rewrite steps by commutativity rules and characterize the set of redex positions of rewrite steps via automata. For this, several conventions, which are going to be introduced now, are useful.

First, we assume $n = \max_{f\in\mathcal{F}} \mathrm{arity}(f) \geq 2$; as, otherwise, one does not have any rewrite step by commutativity rules. And, for the automata characterizing the redex positions, we use DFAs over the signature $\Sigma = \{1, \ldots, n\}$; we put them as *position* automaton.

Definition 3 (position automata). *A DFA $M = \langle Q, \Sigma, \delta, q_0, F\rangle$ is said to be a* position *DFA if $\Sigma = \{1, \ldots, n\}$.*

Now, to work with position DFAs, it is useful to identify each rational term as a complete n-tree, i.e., an infinite tree where all nodes have n-children. Let us assume $\mathrm{arity}(f) = n$ for any $f \in \mathcal{F}$ (including the case $f = \bot$). The rationale for this convention is that we encode $t = f(t_1, \ldots, t_l)$ $(l \leq n)$ over the original signature by $t^\circ = f(t_1^\circ, \ldots, t_l^\circ, t_\bot, \ldots, t_\bot)$, where $t_\bot = \{x_\bot = \bot(x_\bot, \ldots, x_\bot)\}^\star(x_\bot)$ and $x_\bot$ is a special variable reserved for this equation. Thus, *we assume an equation $x_\bot = \bot(x_\bot, \ldots, x_\bot)$ is (implicitly[3]) included to any regular system E.* Moreover, *we also identify each equation $x = z \in E$ where $z \in \mathcal{V} \setminus \mathcal{D}om(E)$ with the equation $x = z(x_\bot, \ldots, x_\bot)$.* Using these conventions, each rational term is identified with a complete n-tree labelled by $f \in \mathcal{F}$ or $z \in \mathcal{V}$.

[3] To ease the readability, however, we omit below the equation $x_\bot = \bot(x_\bot, \ldots, x_\bot)$ if the equation is not necessary, i.e. if there is no equation in E such that its right hand side is a variable or all $f \in \mathcal{F}$ originally have the same arity.

Example 5. Let $\mathcal{F} = \{\mathsf{f}, \mathsf{g}, \bot\}$, $E = \{x = \mathsf{f}(x, y, z), y = \mathsf{g}(y), z = w\}$. We identify E with $E' = \{x = \mathsf{f}(x, y, z), y = \mathsf{g}(y, x_\bot, x_\bot), z = w(x_\bot, x_\bot, x_\bot), x_\bot = \bot(x_\bot, x_\bot, x_\bot)\}$. □

Let $\mathcal{F}_C \subseteq \mathcal{F}$ and $C = \{f(x_1, x_2, x_3, \ldots, x_n) \to f(x_2, x_1, x_3, \ldots, x_n) \mid f \in \mathcal{F}_C\}$. This C is the TRS that we will consider henceforth.

We now show that a DFA that recognized the set of redex positions of a rewrite step can be constructed via canonical regular system that specify that rewrite step.

Definition 4 (canonical DFA). *Let E be a canonical regular system and $W \subseteq \mathcal{D}om(E), x \in \mathcal{D}om(E)$. Then the* canonical *DFA for $\langle E_x, W\rangle$ is a position DFA $\mathcal{M}(E_x, W)$ given by $\langle \mathcal{D}om(E), \Sigma, \delta, x, W\rangle$, where $\delta : \mathcal{D}om(E) \times \Sigma \to \mathcal{D}om(E)$ is defined as $\delta(y, i) = E(y)|_i$.*

Example 6. Let $\mathcal{F} = \{\mathsf{f}, \mathsf{g}\}$, $E = \{x = \mathsf{f}(y, x), y = \mathsf{g}(y, y)\}$ and $F = \{x = \mathsf{f}(x, y), y = \mathsf{g}(y, y)\}$. By applying commutativity rule $\mathsf{f}(x, y) \to \mathsf{f}(y, x)$ to $W = \{x\}$ on E_x we have $s \to^{\Delta} t$ where $\Delta = \{2^n \mid n \geq 0\}$, $s = E^\star(x)$ and $t = F^\star(x)$. Now, the DFA recognizing Δ is obtained as $\mathcal{M}(E_x, W) = \langle \mathcal{D}om(E)(= \{x, y\}), \Sigma(= \{1, 2\}), \delta, x, W\rangle$, where $\delta(z, i) = E(z)|_i$. □

Lemma 4 (redex positions and the language of canonical DFAs). *Let $s \mathrel{\circ\!\!\!\to}^{\Delta} t$ be a rewrite step specified by $\langle E_x, F_x, W\rangle$. Then $\Delta = \mathcal{L}(\mathcal{M}(E_x, W))$.*

Since commutativity rules are symmetric, the rewrite steps by commutativity rules are symmetric. From our definition and the previous lemma, the set of redex positions of the inverse rewrite step also becomes clear.

Lemma 5 (positions of inverse rewrite step). *Let $s \mathrel{\circ\!\!\!\to} t$ be a rewrite step specified by $\langle E_x, F_x, W\rangle$. Then we have a rewrite step $t \mathrel{\circ\!\!\!\to}^{\Lambda} s$ specified by $\langle F_x, E_x, W\rangle$, where $\Lambda = \mathcal{L}(\mathcal{M}(F_x, W))$.*

Now, what is the relation between the set Δ in $s \mathrel{\circ\!\!\!\to}^{\Delta} t$ and the set Λ in $t \mathrel{\circ\!\!\!\to}^{\Lambda} s$? Since these set Δ and Λ are regular sets, the relation should be also characterized via automata. This motives us to define an "inverse" automaton.

The following convention is very useful hereafter: for $i \in \Sigma$, we let $\bar{1} = 2, \bar{2} = 1, \bar{i} = i$ $(3 \leq i \leq n)$.

Definition 5 (inverse automata). *Let $M = \langle Q, \Sigma, \delta, q_0, F\rangle$ be a position DFA. Then we define the* inverse automaton *of M by $M^{-1} = \langle Q, \Sigma, \delta', q_0, F\rangle$ where*

$$\delta'(q, i) = \begin{cases} \delta(q, \bar{i}) & \text{if } q \in F \\ \delta(q, i) & \text{otherwise.} \end{cases}$$

We remark that M^{-1} is a position DFA and $(M^{-1})^{-1} = M$.

First, we consider automata that recognize Δ and Λ of a rewrite step $s \mathrel{\circ\!\!\!\to}^{\Delta} t$ and its inverse $t \mathrel{\circ\!\!\!\to}^{\Lambda} s$ obtained by the triple $\langle E_x, F_x, W\rangle$ that specifies these rewrite step. We show that the automaton for the latter is the inverse of the one for the former.

Lemma 6 (inverse of canonical DFA). *Let $s \mathrel{-\!\!\circ\!\!\!\rightarrow} t$ be a rewrite step specified by $\langle E_x, F_x, W\rangle$. Then we have $\mathcal{M}(E_x, W)^{-1} = \mathcal{M}(F_x, W)$.*

We now show that the inverse operation preserves the equivalence of the languages.

Lemma 7 (language preservation of inverse). *Let M_1, M_2 be position DFAs. If $\mathcal{L}(M_1) = \mathcal{L}(M_2)$ then $\mathcal{L}(M_1^{-1}) = \mathcal{L}(M_2^{-1})$.*

Based on our preparations so far, we are now going to show that regardless of the specification of rewrite steps, inverse rewrite steps are given by reducing the redex positions of the inverse automaton.

Theorem 1 (inverse rewrite steps and inverse automaton). *Let M be a position DFA and suppose $s \mathrel{-\!\!\circ\!\!\!\rightarrow}_C^{\Delta} t$ where $\Delta = \mathcal{L}(M)$. For $\Lambda = \mathcal{L}(M^{-1})$, we have $t \mathrel{-\!\!\circ\!\!\!\rightarrow}_C^{\Lambda} s$.*

Proof. Suppose $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Delta} t$ is specified by $\langle E_x, F_x, W\rangle$. Then, by Lemma 4, we have $\Delta = \mathcal{L}(\mathcal{M}(E_x, W))$. Hence, $\mathcal{L}(M) = \Delta = \mathcal{L}(\mathcal{M}(E_x, W))$ is obtained. Then, by Lemma 7, $\mathcal{L}(M^{-1}) = \mathcal{L}(\mathcal{M}(E_x, W)^{-1})$. On the other hand, by Lemma 5, we have $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma} s$ where $\Gamma = \mathcal{L}(\mathcal{M}(F_x, W))$. Furthermore, by Lemma 6, $\mathcal{M}(E_x, W)^{-1} = \mathcal{M}(F_x, W)$. Thus, $\Lambda = \mathcal{L}(M^{-1}) = \mathcal{L}(\mathcal{M}(E_x, W)^{-1}) = \mathcal{L}(\mathcal{M}(F_x, W)) = \Gamma$. Therefore, from $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma} s$, we obtain $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Lambda} s$. $\square$

Before ending this section, we remark that the results in this section hold not only for the development rewrite step $\mathrel{-\!\!\circ\!\!\!\rightarrow}$ but also for the rewrite step $\rightarrow$, i.e. $s \rightarrow_C^{\Delta} t$ implies $t \rightarrow_C^{\Gamma} s$. The situation, however, becomes different in the next section.

4 Automata for Join of Branching Steps

From this section, we consider automata constructions that arise from branching development rewrite steps, i.e. rewrite steps of the form $t \mathrel{\leftarrow\!\!\!\circ\!\!-} s \mathrel{-\!\!\circ\!\!\!\rightarrow} u$.

The first operation we consider is called *join* of branching steps. Let us explain the intuition of the join of rewrite steps informally. Suppose we have branching rewrite steps from s as $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma} t$ and $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Delta} u$. The join of two rewrite steps expresses the effect of doing these two reductions *simultaneously*. However, this does not mean rewriting all the positions in $\Gamma \cup \Delta$, that is, for $p \in \Gamma \cap \Delta$, we consider applying the commutativity rule twice has an effect same as $s|_p = f(s_1, s_2) \rightarrow f(s_2, s_1) \rightarrow f(s_1, s_2) = s|_p$. That is, we regard that the one rewrite step at $s|_p$ is cancelled by the other. Thus, the join of the redex positions is defined as follows.

Definition 6 (join of position sets). *Let $\Gamma, \Delta \subseteq \mathrm{Pos}(s)$. The* join *of Γ and Δ is defined as $\Gamma \oplus \Delta = \{p \in \Gamma \mid p \notin \Delta\} \cup \{p \in \Delta \mid p \notin \Gamma\}$.*

Example 7. Let $\mathcal{F}_C = \{\mathsf{f}, \mathsf{g}\}$ and $s = \{x = \mathsf{f}(y, z), y = \mathsf{g}(y, w), z = \mathsf{g}(w, z), w = \mathsf{h}(w, w)\}^\star(x)$. Let $\Gamma = \{1^n \mid n \geq 0\}$ and $\Delta = \{2^n \mid n \geq 0\}$. We have $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma \oplus \Delta} t$, where $t = \{x = \mathsf{f}(y, z), y = \mathsf{g}(w, y), z = \mathsf{g}(z, w), w = \mathsf{h}(w, w)\}^\star(x)$. $\square$

We now want to achieve the effect of doing reduction at $\Gamma \oplus \Delta$ on regular systems. Note that two rewrite steps $s \mathrel{\circ\!\!\!\rightarrow}^{\Gamma} t$ and $s \mathrel{\circ\!\!\!\rightarrow}^{\Delta} u$ may be achieved using different regular systems. To synchronize two regular systems, we use the product construction.

We now introduce a notation that is used in the lemma below. Let E, E' be regular systems and $W \subseteq \mathcal{D}om(E), W' \subseteq \mathcal{D}om(E')$. We put $W \oplus W' = (W \times W'^c) \cup (W^c \times W')$. Here, $W^c = \mathcal{D}om(E) \setminus W$ and $W'^c = \mathcal{D}om(E') \setminus W'$.

Lemma 8 (join of branching steps). *Let $E_x, E'_{x'}$ be regular representations of s. Let $s \mathrel{\circ\!\!\!\rightarrow}^{\Gamma} t$ ($s \mathrel{\circ\!\!\!\rightarrow}^{\Delta} u$) be the rewrite step obtained by applying the rewrite rules on $W \subseteq \mathcal{D}om(E)$ of E_x ($W' \subseteq \mathcal{D}om(E')$ of $E'_{x'}$, respectively). Then, $(E \times E')_{\langle x,x' \rangle}$ is a regular representation of s, and by applying the rewrite rules on $W \oplus W'$ of $(E \times E')_{\langle x,x' \rangle}$, one obtains a rewrite step $s \mathrel{\circ\!\!\!\rightarrow}^{\Gamma \oplus \Delta} v$ for some v. (Hence, $\Gamma \oplus \Delta = \mathcal{L}(\mathcal{M}((E \times E')_{\langle x,x' \rangle}, W \oplus W'))$.)*

The previous lemma motivates us to introduce the following automata construction.

Definition 7 (join automata). *We define the* join automaton $M_1 \oplus M_2$ *of two position DFAs $M_1 = \langle Q_1, \Sigma, \delta_1, q_1, F_1 \rangle$ and $M_2 = \langle Q_2, \Sigma, \delta_2, q_2, F_2 \rangle$ as follows: $M_1 \oplus M_2 = \langle Q_1 \times Q_2, \Sigma, \delta, \langle q_1, q_2 \rangle, F_1 \oplus F_2 \rangle$, where*

- *δ is given like this: $\delta(\langle x, y \rangle, i) = \langle \delta_1(x, i), \delta_2(y, i) \rangle$ and*
- *$F_1 \oplus F_2 = \{\langle x, y \rangle \mid x \in F_1, y \in Q_2 \setminus F_2\} \cup \{\langle x, y \rangle \mid x \in Q_1 \setminus F_1, y \in F_2\}$.*

Next lemmas are easily obtained.

Lemma 9 (join of canonical DFAs). *Let $E_x, E'_{x'}$ be regular representations of s, $W \subseteq \mathcal{D}om(E)$, and $W' \subseteq \mathcal{D}om(E')$. Then, $\mathcal{M}(E_x, W) \oplus \mathcal{M}(E'_x, W') = \mathcal{M}((E \times E')_{\langle x,x' \rangle}, W \oplus W')$.*

Lemma 10 (language preservation of join). *Suppose that M_1, M_2, M'_1, M'_2 are position DFAs. If $\mathcal{L}(M_1) = \mathcal{L}(M'_1)$ and $\mathcal{L}(M_2) = \mathcal{L}(M'_2)$ then $\mathcal{L}(M_1 \oplus M_2) = \mathcal{L}(M'_1 \oplus M'_2)$.*

We now arrive the main theorem of this section.

Theorem 2 (join rewrite steps and join automata). *Let M_1, M_2 be position DFAs. Suppose $s \mathrel{\circ\!\!\!\rightarrow}^{\Gamma}_{C} t$ and $s \mathrel{\circ\!\!\!\rightarrow}^{\Delta}_{C} u$, where $\Gamma = \mathcal{L}(M_1)$ and $\Delta = \mathcal{L}(M_2)$. Then, $s \mathrel{\circ\!\!\!\rightarrow}^{\Gamma \oplus \Delta}_{C} v$ and $\Gamma \oplus \Delta = \mathcal{L}(M_1 \oplus M_2)$ for some v.*

Proof. Suppose that the rewrite step $s \mathrel{\circ\!\!\!\rightarrow}^{\Gamma} t$ ($s \mathrel{\circ\!\!\!\rightarrow}^{\Delta} u$) is obtained by applying the rewrite rules on W of E_x (W' of $E'_{x'}$, respectively). Then $\Gamma = \mathcal{L}(M_1) = \mathcal{L}(\mathcal{M}(E_x, W))$ and $\Delta = \mathcal{L}(M_2) = \mathcal{L}(\mathcal{M}(E'_{x'}, W'))$. Then, by Lemma 10, we have $\mathcal{L}(M_1 \oplus M_2) = \mathcal{L}(\mathcal{M}(E_x, W) \oplus \mathcal{M}(E'_{x'}, W'))$. By Lemmas 8 and 9, $s \mathrel{\circ\!\!\!\rightarrow}^{\Gamma \oplus \Delta} v$ and $\Gamma \oplus \Delta = \mathcal{L}(\mathcal{M}((E \times E')_{\langle x,x' \rangle}, W \oplus W')) = \mathcal{L}(\mathcal{M}(E_x, W) \oplus \mathcal{M}(E'_{x'}, W')) = \mathcal{L}(M_1 \oplus M_2)$. □

Remark 2. For branching (standard) steps $s \rightarrow^{\Gamma}_{C} t_1$ and $s \rightarrow^{\Delta}_{C} t_2$, we obtain $s \mathrel{\circ\!\!\!\rightarrow}^{\Gamma \oplus \Delta}_{C} v$, as $s \rightarrow_C t_i$ implies $s \mathrel{\circ\!\!\!\rightarrow}_C t_i$. However, because the employed rules in $s \rightarrow^{\Gamma}_{C} t_1$ and $s \rightarrow^{\Delta}_{C} t_2$ may be different, it is not always the case $s \rightarrow^{\Gamma \oplus \Delta}_{C} v$. This is why we had to introduce the development rewrite step $\mathrel{\circ\!\!\!\rightarrow}$.

5 Automata for Difference of Branching Steps

Suppose that we have $s \xrightarrow{\ominus}{}^{\Gamma}_{C} t$, $s \xrightarrow{\ominus}{}^{\Delta}_{C} u$ and $s \xrightarrow{\ominus}{}^{\Gamma\oplus\Delta}_{C} v$. Then, naturally there would be a rewrite step that will close the gap between t and v (u and v)—we will call rewrite steps such as $t \xrightarrow{\ominus} v$ and $u \xrightarrow{\ominus} v$ 'difference' of that branching rewrite steps. Below we present an automata construction that capture taking the difference of that branching rewrite steps.

Below, we put $(f(t_1, t_2, t_3, \ldots t_n))^C = f(t_2, t_1, t_3, \ldots, t_n)$.

Lemma 11 (difference of branching steps). *Let $E_x, E'_{x'}$ be regular representations of s. Let $s \xrightarrow{\ominus}{}^{\Gamma} t$ ($s \xrightarrow{\ominus}{}^{\Delta} u$) be the rewrite step obtained by applying the rewrite rules on $W \subseteq \mathcal{D}om(E)$ of E_x ($W' \subseteq \mathcal{D}om(E')$ of $E'_{x'}$, respectively). Suppose $s \xrightarrow{\ominus}{}^{\Gamma\oplus\Delta} v$.*

1. *Let $F = \{\langle y, y'\rangle = w^C \mid \langle y, y'\rangle = w \in E \times E', y \in W\} \cup \{\langle y, y'\rangle = w \in E \times E' \mid y \notin W\}$. Then, $F_{\langle x,x'\rangle}$ is a regular representation of t and one obtains a rewrite step $t \xrightarrow{\ominus} v$ by applying the rewrite rules on $\mathcal{D}om(E) \times W'$ of $F_{\langle x,x'\rangle}$.*
2. *Let $F' = \{\langle y, y'\rangle = w^C \mid \langle y, y'\rangle = w \in E \times E', y' \in W'\} \cup \{\langle y, y'\rangle = w \in E \times E' \mid y' \notin W'\}$. Then, $F'_{\langle x,x'\rangle}$ is a regular representation of u and one obtains a rewrite step $u \xrightarrow{\ominus} v$ by applying the rewrite rules on $W \times \mathcal{D}om(E')$ of $F'_{\langle x,x'\rangle}$.*

The characterization of the previous lemma motivates us to define the difference automata as follows.

Definition 8 (difference automata). *Let $M_1 = \langle Q_1, \Sigma, \delta_1, q_1, F_1\rangle$, $M_2 = \langle Q_2, \Sigma, \delta_2, q_2, F_2\rangle$ be position DFAs. We define the* difference automaton *by $M_2 \setminus M_1 = \langle Q_1 \times Q_2, \Sigma, \eta, \langle q_1, q_2\rangle, Q_1 \times F_2\rangle$, where η is given like this:*

$$\eta(\langle x, y\rangle, i) = \begin{cases} \langle \delta_1(x, \bar{i}), \delta_2(y, \bar{i})\rangle & \text{if } x \in F_1 \\ \langle \delta_1(x, i), \delta_2(y, i)\rangle & \text{otherwise.} \end{cases}$$

The next lemma is shown using Lemma 11.

Lemma 12 (difference of canonical DFAs). *Let $s \xrightarrow{\ominus}{}^{\Gamma} t$ ($s \xrightarrow{\ominus}{}^{\Delta} u$) be obtained by applying the rewrite rules on W of E_x (on W' of $E'_{x'}$, respectively). Suppose $s \xrightarrow{\ominus}{}^{\Gamma\oplus\Delta} v$. Then (1) $t \xrightarrow{\ominus}{}^{\Lambda} v$ where $\Lambda = \mathcal{L}(\mathcal{M}(E'_{x'}, W') \setminus \mathcal{M}(E_x, W))$, and (2) $u \xrightarrow{\ominus}{}^{\Pi} v$ where $\Pi = \mathcal{L}(\mathcal{M}(E_x, W) \setminus \mathcal{M}(E'_{x'}, W'))$.*

Lemma 13 (language preservation of difference). *Let M_1, M'_1, M_2, M'_2 be position DFAs such that $\mathcal{L}(M_1) = \mathcal{L}(M'_1)$ and $\mathcal{L}(M_2) = \mathcal{L}(M'_2)$. Then, $\mathcal{L}(M_2 \setminus M_1) = \mathcal{L}(M'_2 \setminus M'_1)$.*

Thus, we are ready to show that the difference of branching rewrite steps is characterized by the difference automata.

Theorem 3 (difference rewrite steps and difference automata). *Let M_1, M_2 be position DFAs. Let $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma}_{C} t$ and $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Delta}_{C} u$, where $\Gamma = \mathcal{L}(M_1)$ and $\Delta = \mathcal{L}(M_2)$. Suppose $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma \oplus \Delta}_{C} v$. Then, (1) $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Lambda}_{C} v$, where $\Lambda = \mathcal{L}(M_2 \setminus M_1)$, and (2) $u \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Lambda'}_{C} v$, where $\Lambda' = \mathcal{L}(M_1 \setminus M_2)$.*

Proof. We here only show (1), as (2) can be shown in the symmetric way. Suppose that the rewrite step $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma} t$ ($s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Delta} u$) is obtained by applying the rewrite rules on W of E_x (W' of $E'_{x'}$, respectively). Then $\Gamma = \mathcal{L}(M_1) = \mathcal{L}(\mathcal{M}(E_x, W))$ and $\Delta = \mathcal{L}(M_2) = \mathcal{L}(\mathcal{M}(E'_{x'}, W'))$. Then, it follows from Lemma 13 that $\mathcal{L}(M_2 \setminus M_1) = \mathcal{L}(\mathcal{M}(E'_{x'}, W') \setminus \mathcal{M}(E_x, W))$. By Lemma 12, $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Lambda} v$ by taking $\Lambda = \mathcal{L}(\mathcal{M}(E'_{x'}, W') \setminus \mathcal{M}(E_x, W)) = \mathcal{L}(M_2 \setminus M_1)$. $\square$

6 Closure under Equivalence

In this section, we give an application of the results in previous three sections. Namely, we show that development rewrite step $\mathrel{-\!\!\circ\!\!\!\rightarrow}_{C}$ is closed under taking equivalence. It is clear from the definition that $\mathrel{-\!\!\circ\!\!\!\rightarrow}_{C}$ is reflexive, and in Theorem 1 we have already shown that $\mathrel{-\!\!\circ\!\!\!\rightarrow}_{C}$ is symmetric. Thus, only transitivity is yet to be shown.

We need one lemma for this.

Lemma 14. *For any position DFAs M_1, M_2, we have $\mathcal{L}((M_2 \setminus {M_1}^{-1}) \setminus M_1) = \mathcal{L}(M_2)$.*

Theorem 4 (merging of consecutive steps). *Let M_1, M_2 be position DFAs. Let $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Delta}_{C} t$ and $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma}_{C} u$, where $\Delta = \mathcal{L}(M_1)$ and $\Gamma = \mathcal{L}(M_2)$. Then, $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Lambda}_{C} u$, where $\Lambda = \mathcal{L}((M_2 \setminus M_1^{-1}) \oplus M_1)$.*

Proof. From $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Delta} t$ and Theorem 1, we have $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Delta'} s$, where $\Delta' = \mathcal{L}(M_1^{-1})$. Thus, from $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Delta'} s$ and $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma} u$, we obtain by Theorem 2 that $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Delta' \oplus \Gamma} v$ for some v. Furthermore, $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Pi} v$ by the Theorem 3, where $\Pi = \mathcal{L}(M_2 \setminus {M_1}^{-1})$. Now we have $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Pi} v$ and $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Delta} t$. Thus, from Theorem 2, we have $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Lambda} u'$ for some u', where $\Lambda = \Pi \oplus \Delta = \mathcal{L}((M_2 \setminus M_1^{-1}) \oplus M_1)$. Furthermore, we have $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma'} u'$ by Theorem 3, where $\Gamma' = \mathcal{L}((M_2 \setminus {M_1}^{-1}) \setminus M_1)$. From Lemma 14, $\Gamma' = \mathcal{L}((M_2 \setminus {M_1}^{-1}) \setminus M_1) = \mathcal{L}(M_2) = \Gamma$. Thus, since we have $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma} u$ by our assumption, we obtain $u = u'$ from $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Gamma'} u'$. As we have $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Lambda} u'$, we conclude $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^{\Lambda} u$. $\square$

The following is an immediate corollary of Theorems 1 and 4.

Corollary 1 (closure under equivalence). *Equivalence closure of development rewrite steps is identical to a single development rewrite step in rational term rewriting of commutativity rules, i.e. $\mathrel{\leftarrow\!\!\circ\!\!\!\rightarrow}^{*}_{C} = \mathrel{-\!\!\circ\!\!\!\rightarrow}_{C}$ in rational term rewriting for any set C of commutativity rules.*

7 Conclusion

We have studied development rewrite steps $\mathrel{-\!\!\circ\!\!\!\rightarrow}_C$ of rational term rewriting by commutativity rules C, where each rewrite step $s \mathrel{-\!\!\circ\!\!\!\rightarrow}_C^{\Gamma} t$ is specified by a regular set Γ of positions (hence by a finite automaton) in the rational term s. We have shown the inverse automata construction $()^{-1}$ such that $s \mathrel{-\!\!\circ\!\!\!\rightarrow}_C^{\mathcal{L}(M)} t$ give rise to $t \mathrel{-\!\!\circ\!\!\!\rightarrow}_C^{\mathcal{L}(M^{-1})} s$. We have also given the constructions of join $M_1 \oplus M_2$ and difference $M_1 \setminus M_2$ of automata M_1 and M_2 specifying branching steps $s \mathrel{-\!\!\circ\!\!\!\rightarrow}_C^{\mathcal{L}(M_1)} t_1$ and $s \mathrel{-\!\!\circ\!\!\!\rightarrow}_C^{\mathcal{L}(M_2)} t_2$. Then, consecutive steps $s \mathrel{-\!\!\circ\!\!\!\rightarrow}_C^{\mathcal{L}(M_1)} t \mathrel{-\!\!\circ\!\!\!\rightarrow}_C^{\mathcal{L}(M_2)} u$ give rise to $s \mathrel{-\!\!\circ\!\!\!\rightarrow}_C^{\mathcal{L}(M')} u$ with $M' = (M_2 \setminus {M_1}^{-1}) \oplus M_1$. As a corollary, it has been shown that the equivalence closure $\mathrel{\leftarrow\!\!\circ\!\!\!\rightarrow}_C^*$ of development rewrite steps is identical to a single development rewrite step $\mathrel{-\!\!\circ\!\!\!\rightarrow}_C$ for any set C of commutativity rules.

A possible future work would be the commutative unification in the setting of rational term rewriting. It would be also an interesting question how one can obtain the automata constructions for showing reversibility (i.e. $s \mathrel{-\!\!\circ\!\!\!\rightarrow}^* t$ implies $t \mathrel{-\!\!\circ\!\!\!\rightarrow}^* s$) of associative-commutative rational term rewriting. Another possible future work would be to generalize our constructions to deal with any *flat* rules.

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