Termination of Rule-Based Calculi for Uniform Semi-Unification

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Abstract. Uniform semi-unification is a generalization of unification; its efficient algorithms have been extensively studied in (Kapur et al., 1994) and (Oliart&Snyder, 2004). For (uniform) semi-unification, several variants of rule-based calculi are known. But, some of these calculi in the literature lack the termination property, i.e. not all derivations are terminating. We revisit symbolic semi-unification whose solvability coincides with that of uniform semi-unification. We give an abstract criterion of the strategy on which a general rule-based calculus for symbolic semi-unification terminates. Based on this, we give an alternative and robust correctness proof of a rule-based uniform semi-unification algorithm.

Keywords: Semi-Unification, Rule-Based Calculi, Termination

1 Introduction

Describing algorithms using rule-based calculi is often useful to present algorithms abstractly and to study the correctness of algorithms by separating the issue from those of the search strategy and/or efficiency. For many algorithms, rule-based calculi have been widely adapted and commonly used to extend or modify the algorithms; well-known such examples are the unification algorithm and the Knuth-Bendix completion algorithm (see e.g. [1]).

Semi-unification is a generalization of unification. Its application includes non-termination proving of term rewriting systems [3, 8, 16] and polymorphic type inference problems of ML languages [7, 11]; it is also related to some problems in proof theory [17] and in computational linguistics [2]. Like unification, if a semi-unifier exists, there exists a most general semi-unifier [7, 10, 17]. Unlike unification, however, semi-unification is undecidable in general [10]. Hence, many decidable classes of semi-unification have been studied: uniform semi-unification [4, 8, 15, 17–19], acyclic semi-unification [11, 14], left-linear semi-unification [6, 9], quasi-monadic semi-unification [13] and semi-unification in two variables [12].

Decidability of uniform semi-unification has been shown in various articles almost at the same time [4, 5, 8, 12, 17]. Efficient algorithms for uniform semi-unification have been extensively studied in [8, 15]. Like with unification, working on graphs is mandatory in order to give efficient algorithms, and thus these best efficient algorithms are based on graphs. The basis of correctness of these algorithms, however, is given by a simple version of the algorithms given in [8] that can be almost adapted as a rule-based calculus (not on graphs but on terms). In [15], a variant of this calculus are adapted as a rule-based calculus but no correctness proof is presented.

In contrast to similar rule-based calculi for the unification, some of such calculi in the literature lack the termination property, i.e. not all derivations are terminating. It is mentioned in [8] that “it can be shown that Algorithm A-1 will always terminate either reporting failure or semi-unifiability;...” without a proof. But, actually, derivations in their rule-based calculus terminate only if some suitable interpretation or derivation strategy is assumed. The calculus given in [15] is more flexible than that of [8], and it is not terminating either.

It may be thought that giving a terminating variant of these calculi is easy. However, it is, at least, difficult to adapt a termination proof similar to the one applied to calculi for unification for the following reasons: (1) equations in “solved” form may later be changed to “unsolved”, and (2) the multiset of the (extended) variables in the equations may increase. In fact, as pointed out in [13], it seems not easy to give a well-founded ordering that guarantees the termination of such calculi.

In this paper, we revisit symbolic semi-unification, whose solvability coincides with that of uniform semi-unification. We present a new characterization of symbolic semi-unification, which reinforces the correspondence between these presentations of uniform semi-unification. We give an abstract criterion of the strategy on which a general rule-based calculus for symbolic semi-unification terminates. In this way, we give an alternative and robust proof for the correctness of a rule-based uniform semi-unification algorithm.

2 Preliminaries

We denote sets of *arity-fixed function symbols* and *variables* by $\mathcal{F}$ and $\mathcal{V}$, respectively. The set of *terms* is denote by $\mathrm{T}(\mathcal{F},\mathcal{V})$. The set of variables in an object α, which may be a term, etc. is denoted by $\mathcal{V}(\alpha)$. A *context* is a term in $\mathrm{T}(\mathcal{F}\cup\{\Box\},\mathcal{V})$ containing a single occurrence of $\Box$, which is a special constant not contained in $\mathcal{F}$. A term obtained by replacing $\Box$ in a context C with a term t is denoted by $C[t]$. A term s is a *subterm* of a term t (written as $s \trianglelefteq t$) if $t = C[s]$ for some context C. A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathrm{T}(\mathcal{F},\mathcal{V})$ with the finite domain $\mathrm{dom}(\sigma) = \{x \mid x \neq \sigma(x)\}$. Substitutions are homomorphically extended to mappings over $\mathrm{T}(\mathcal{F},\mathcal{V})$. We write $\sigma(t)$ as $t\sigma$. A substitution σ with $\mathrm{dom}(\sigma) = \{x_1,\ldots,x_n\}$ and $\sigma(x_i) = t_i$ is denoted by $\{x_1 := t_1,\ldots,x_n := t_n\}$.

We denote an *equation* by $s \approx t$ which is indistinguished from $t \approx s$ and an *inequation* by $s \leqslant t$. We also consider indexed inequations of the form $s \leqslant_i t$ where the index i ranges over $1,\ldots,k$. Let $E = \{s_i \circ_i t_i \mid 1 \le i \le n, \circ_i \in \{\approx, \leqslant_1, \ldots, \leqslant_k\}\}$ be a set of equations and indexed inequations. Then E is said to be *semi-unifiable* if there exists a substitution $\tau,\rho_1,\ldots,\rho_k$ such that $\tau(s_i) = \tau(t_i)$

for all $s_i \approx t_i \in E$ and $\rho_j(\tau(s_i)) = \tau(t_i)$ for all $s_i \leqslant_j t_i \in E$; the substitution τ is called a *semi-unifier* of E and the substitutions $\rho_1, \ldots, \rho_k$ are called *residual substitutions* of the semi-unifier τ. A *semi-unification problem* is a problem to ask whether there is a semi-unifier for a given set of equations and (indexed) inequations. A semi-unification problem is said to be *uniform* if $k = 1$ (i.e. the index of inequations is unique); when we think of a uniform semi-unification problem the index of inequations will be omitted. Any (uniform) semi-unification problem E can be reduced to a (uniform) semi-unification problem without equations by replacing $s_i \approx t_i \in E$ with $z_i \leqslant s_i, z_i \leqslant t_i$ using a fresh variable z_i. Thus, one can assume w.l.o.g. that any semi-unification problem deals with only the set of inequations.

Example 1. Let $E = \{\mathsf{f}(\mathsf{h}(y), x) \leqslant \mathsf{f}(x, \mathsf{h}(\mathsf{h}(y)))\}$. Take $\sigma = \{x := \mathsf{h}(y')\}$ and $\rho_1 = \{y := y', y' := \mathsf{h}(y)\}$, where y' is a fresh variable. We have $\mathsf{f}(\mathsf{h}(y), x)\sigma\rho_1 = \mathsf{f}(\mathsf{h}(y), \mathsf{h}(y'))\rho_1 = \mathsf{f}(\mathsf{h}(y'), \mathsf{h}(\mathsf{h}(y))) = \mathsf{f}(x, \mathsf{h}(\mathsf{h}(y)))\sigma$. Thus E is semi-unifiable and σ is a semi-unifier. Note here that $\mathsf{f}(\mathsf{h}(y), x)$ and $\mathsf{f}(x, \mathsf{h}(\mathsf{h}(y)))$ are not unifiable.

3 Symbolic Semi-Unification

In this section, we introduce a notion of symbolic[1] semi-unification. The notion is based on the idea of syntactically representing the substitution ρ of the identity $\rho(\tau(s)) = \tau(t)$ expressing semi-unifiability. This idea goes back to [8]. Our presentation mostly follows a nicer formulation given in [13]. In the literature, various symbols are used as the "place holder" for ρ; we here use ∇ as it is clearly distinguished from substitutions denoted by ρ, τ, σ, etc.

Definition 2 (symbol ∇, ∇-variables, ∇-terms, operator ∇).

1. *We use a* unary special function symbol ∇ *which is supposed to be not contained in* $\mathcal{F}$.
2. *We define* ∇-terms *as follows: (i)* $\nabla^i(x)$ *where* $x \in \mathcal{V}$ *and* $i \geq 0$ *are* ∇*-terms where* $\nabla^i(x)$ *abbreviates* $\overbrace{\nabla(\cdots\nabla}^{i\text{-times}}(x)\cdots)$*; (ii) if* $t_1, \ldots, t_n$ *are* ∇*-terms then* $f(t_1, \ldots, t_n)$ *is a* ∇*-term for any* $f \in \mathcal{F}$ *of arity* n*. Equations of* ∇*-terms are said to be* ∇-equations.
3. ∇*-terms of the form* $\nabla^i(x)$ $(i \geq 0)$ *are called* ∇-variables. *We denote* $\nabla^i(x)$ *by* x^i*. Hence* $x^0 = x$*,* $\nabla(x^i) = x^{i+1}$*, and* $x^j \trianglelefteq x^i$ *for all* $j \leq i$*. The sets of* ∇*-variables and* ∇*-terms are denoted by* $\mathcal{V}^*$ *and* $\mathrm{T}(\mathcal{F}, \mathcal{V}^*)$*, respectively. The set of* ∇*-variables in an object* α *is denoted by* $\mathcal{V}^*(\alpha)$*.*
4. *We define a* unary operation ∇ *on* ∇*-terms recursively as follows:* $\nabla(x^i) = x^{i+1}$*;* $\nabla(f(t_1, \ldots, t_n)) = f(\nabla(t_1), \ldots, \nabla(t_n))$*.*

Example 3. A ∇-term $t = \mathsf{f}(x, \mathsf{g}(\nabla(\nabla(y))))$ may be also written as $\mathsf{f}(x, \mathsf{g}(\nabla^2(y)))$ or $\mathsf{f}(x, \mathsf{g}(y^2))$. The set of ∇-variables in t is $\mathcal{V}^*(t) = \{x, y, \nabla(y), \nabla(\nabla(y))\} = \{x, y, y^1, y^2\}$. We have $\nabla(t) = \nabla(\mathsf{f}(x, \mathsf{g}(\nabla^2(y)))) = \mathsf{f}(\nabla(x), \nabla(\mathsf{g}(\nabla^2(y)))) = \mathsf{f}(\nabla(x), \mathsf{g}(\nabla^3(y))) = \mathsf{f}(x^1, \mathsf{g}(y^3))$ and $\nabla^2(t) = \nabla(\nabla(t)) = \mathsf{f}(x^2, \mathsf{g}(y^4))$.

[1] The name "symbolic" is from [19].

Contexts over ∇-terms and the subterm relation on ∇-terms are defined similarly to the usual contexts and subterms. We define ∇-substitutions below.

Definition 4 (∇-substitution).

1. *A* ∇-substitution *is a partial mapping σ from $\mathcal{V}^*$ to $\mathrm{T}(\mathcal{F},\mathcal{V}^*)$ such that (i) the domain* $\mathrm{dom}(\sigma)$ *of σ is finite; (ii) for each $x \in \mathcal{V}$ there exists at most one i such that $x^i \in \mathrm{dom}(\sigma)$; (iii) for each $x^i, y^j \in \mathrm{dom}(\sigma)$, $y^j \not\trianglelefteq \sigma(x^i)$.*
2. *The application $\sigma(t)$ of a ∇-substitution σ to a ∇-term t is recursively defined as follows: $\sigma(y^j) = y^j$ if $y^i \notin \mathrm{dom}(\sigma)$ for any $i \leq j$; $\sigma(y^j) = \nabla^{j-i}(\sigma(y^i))$ if $y^i \in \mathrm{dom}(\sigma)$ for some $i \leq j$; $\sigma(f(t_1,\ldots,t_n)) = f(\sigma(t_1),\ldots,\sigma(t_n))$.*
3. *The many-time application $\sigma^*(t)$ of a ∇-substitution σ to a ∇-term t is defined as follows: $\sigma^*(t) = t$ if $x^i \not\trianglelefteq t$ for any $x^i \in \mathrm{dom}(\sigma)$; $\sigma^*(t) = \sigma^*(\sigma(t))$ otherwise.*

We write $\sigma(t)$ as $t\sigma$ and $\sigma^*(t)$ as $t\sigma^*$; in particular, we write $\sigma^*(\nabla(t))$ as $\nabla(t)\sigma^*$. The notion of many-time application of a ∇-substitution to a ∇-term is used to give an invariance of derivation for symbolic semi-unification.

Example 5. The partial mapping $\sigma = \{x := \mathsf{a}, x^1 := \mathsf{b}\}$ is not a ∇-substitution, as the σ does not satisfy the condition (ii). Neither is the partial mapping $\sigma = \{x^1 := \mathsf{f}(y^1), y^1 := \mathsf{b}\}$, as the σ does not satisfy the condition (iii). The partial mapping $\sigma = \{x^1 := y, y^1 := \mathsf{f}(z^2)\}$ is a ∇-substitution, and we have $\sigma(y^3) = \nabla^2(\sigma(y^1)) = \nabla^2(\mathsf{f}(z^2)) = \mathsf{f}(z^4)$ and $\sigma^*(x^2) = \sigma^*(\nabla(y)) = \sigma^*(y^1) = \mathsf{f}(z^2)$.

It may be not so obvious from the definition that $t\sigma^*$ is always well-defined; however, this can be derived from our definition of ∇-substitutions.

Lemma 6. *For any ∇-substitution σ and ∇-term t, $t\sigma^*$ is well-defined.*

Proof. Define, for each ∇-variable x^j, $w(x^j) = \max\{j - i + 1, 0\}$ if $x^i \in \mathrm{dom}(\sigma)$ for some i, and $w(x^j) = 0$ otherwise. Let $\mathcal{W}(t)$ be the multiset of weight of ∇-variables in a ∇-term t. Then if $t \neq t\sigma$ then $\mathcal{W}(t) \gg \mathcal{W}(t\sigma)$, where $\gg$ is the multiset extension (e.g. [1]) of the natural order $>$ on the set of natural numbers. The claim follows from the well-foundedness of $\gg$. □

Lemma 7. *For any ∇-term t and ∇-substitution σ, $\nabla(t)\sigma^* = \nabla(t\sigma^*)\sigma^*$.*

Proof. Let $\mathcal{R} = \{u_g \to v_g \mid u := v \in \sigma\}$ be a TRS (see e.g. [1]), where $()_g$ replaces each variable with a distinct constant. The claim follows from completeness of the TRS $\mathcal{R} \cup \{\nabla(f(x_1,\ldots,x_n)) \to f(\nabla(x_1),\ldots,\nabla(x_n)) \mid f \in \mathcal{F}\}$. □

We now introduce a notion of symbolic semi-unification.

Definition 8 (symbolic semi-unification). *For a set E of ∇-equations, a* semi-unifier *of E is a ∇-substitution σ such that $s\sigma^* = t\sigma^*$ for all $s \approx t \in E$; if E has a semi-unifier, E is said to be* semi-unifiable. *A* symbolic semi-unification problem *asks whether there exists a semi-unifier for a given set of ∇-equations.*

The next lemma is shown using Lemma 7.

Lemma 9 (semi-unifiability is closed under ∇). *Let σ be a ∇-substitution and s, t be ∇-terms. If $s\sigma^* = t\sigma^*$ then $\nabla(s)\sigma^* = \nabla(t)\sigma^*$.*

Remark 10. In contrast to Lemma 9, $s\sigma = t\sigma$ does not necessary imply $\nabla(s)\sigma = \nabla(t)\sigma$: Let $\sigma = \{x^2 := \mathsf{f}(x)\}$, $s = x^3$ and $t = \mathsf{f}(x^1)$. Then $s\sigma = \mathsf{f}(x^1) = t\sigma$, but $\nabla(s)\sigma = x^4\sigma = \mathsf{f}(x^2) \neq \mathsf{f}(\mathsf{f}(x)) = \mathsf{f}(x^2)\sigma = \nabla(t)\sigma$.

The notions of ∇-equality and inconsistency were introduced in [8].

Definition 11 (∇-equality). *For a set E of ∇-equations, the* ∇-equality generated by E*, denoted by $\approx_E$, is the smallest equivalence relation such that (i) $s \approx_E t$ for any $s \approx t \in E$, (ii) $s \approx_E t$ implies $\nabla(s) \approx_E \nabla(t)$, and (iii) for any $f \in \mathcal{F}$, $f(s_1, \ldots, s_n) \approx_E f(t_1, \ldots, t_n)$ iff, for any $i = 1, \ldots, n$, $s_i \approx_E t_i$ holds.*

Definition 12 (inconsistency). *A set E of ∇-equations is* inconsistent *if either (i) $x^i \approx_E s$ with $x^i \trianglelefteq s \notin \mathcal{V}^*$, or (ii) $f(s_1, \ldots, s_m) \approx_E g(t_1, \ldots, t_n)$ with $f \neq g$ for some $f, g \in \mathcal{F}$. Furthermore, E is* consistent *if it is not inconsistent.*

The next lemma will be used heavily in our proof.

Lemma 13. *Let E be a set of ∇-equations. Suppose E is semi-unifiable and let σ be a semi-unifier of E. Then for any ∇-terms u, v, $u \approx_E v$ implies $u\sigma^* = v\sigma^*$.*

Proof. By induction on the derivation of $u \approx_E v$ using Lemma 9. ◻

4 Symbolic Semi-Unification and Semi-Unification

In this section, we show the equivalence between the consistency and the symbolic semi-unifiability of ∇-equations and the semi-unifiability of the corresponding inequations. Thus, we extend and give a rigorous proof of a result of [8]. A part of the proof will be postponed until Section 6.

We first introduce interpretations of ∇-terms, a key notion of our proof.

Definition 14 (interpretation). *Let τ, ρ be substitutions. An* interpretation *$[\![t]\!]^\tau_\rho \in \mathrm{T}(\mathcal{F}, \mathcal{V})$ of $t \in \mathrm{T}(\mathcal{F}, \mathcal{V}^*)$ is given by $[\![x^i]\!]^\tau_\rho = \rho^i(\tau(x))$; $[\![f(s_1, \ldots, s_n)]\!]^\tau_\rho = f([\![s_1]\!]^\tau_\rho, \ldots, [\![s_n]\!]^\tau_\rho)$.*

Lemma 15. *Let τ, ρ be substitutions. (1) For $t \in \mathrm{T}(\mathcal{F}, \mathcal{V}^*)$, $[\![\nabla(t)]\!]^\tau_\rho = \rho([\![t]\!]^\tau_\rho)$. (2) For $t \in \mathrm{T}(\mathcal{F}, \mathcal{V})$, $[\![t]\!]^\tau_\rho = \tau(t)$.*

Proof. By induction on t. ◻

A solution $\sigma_{\mathsf{u}}, \sigma_{\mathsf{m}}$ of a uniform semi-unification problem can be obtained from a solution σ of symbolic semi-unification problem as follows.

Definition 16 ([8]). *Let σ be a ∇-substitution. Then we define its unification part σ_{u} and matching part σ_{m} as below.*

1. *Let $X_0 = \{x \in \mathcal{V} \mid x \in \mathrm{dom}(\sigma)\}$, $X_1 = \{x^i \mid x^i \in \mathrm{dom}(\sigma), i > 0\}$, and $X_2 = (\bigcup\{\mathcal{V}^*(x^i \approx \sigma(x^i)) \mid x^i \in \mathrm{dom}(\sigma)\}) \setminus (\mathcal{V} \cup X_1)$.*

2. *Prepare fresh distinct variables for each* $x^i \in X_2$ *and let* φ *be a mapping that assigns to each* $x^i \in X_2$ *the corresponding fresh variable; let* φ *be recursively extended as* $\varphi(y) = y$; $\varphi(f(t_1, \ldots, t_n)) = f(\varphi(t_1), \ldots, \varphi(t_n))$.
3. *For each* $x^i \in X_1$, *put* $\sigma_{\mathsf{m}}(x^{i-1}) = \varphi(\sigma(x^i))$ *if* $i = 1$, *and* $\sigma_{\mathsf{m}}(\varphi(x^{i-1})) = \varphi(\sigma(x^i))$ *otherwise. For each* $x^i \in X_2$, *put* $\sigma_{\mathsf{m}}(x^{i-1}) = \varphi(x^i)$ *if* $i = 1$, *and* $\sigma_{\mathsf{m}}(\varphi(x^{i-1})) = \varphi(x^i)$ *otherwise.*
4. *For each* $x \in X_0$, *put* $\sigma_{\mathsf{u}}(x) = \varphi(\sigma(x))$.

Example 17. Let $\sigma = \{x := \mathsf{f}(y, z^1), y^1 := \mathsf{f}(y, z), z^3 := \mathsf{f}(z^1, w^2)\}$. Then $X_0 = \{x\}$, $X_1 = \{y^1, z^3\}$ and $X_2 = \{z^1, z^2, w^1, w^2\}$. Put $\varphi = \{z^1 := z', z^2 := z'', w^1 := w', w^2 := w''\}$. Then we have $\sigma_{\mathsf{m}} = \{y := \mathsf{f}(y, z), z'' := \mathsf{f}(z', w''), z := z', z' := z'', w := w', w' := w''\}$ and $\sigma_{\mathsf{u}} = \{x := \mathsf{f}(y, z')\}$.

Lemma 18. *For any* ∇*-substitution* σ, σ_{u} *and* σ_{m} *are well-defined substitutions.*

Lemma 19. *(1)* $\sigma^i_{\mathsf{m}}(x) = \varphi(x^i)$ *for* $x^i \in X_2$. *(2)* $\sigma^i_{\mathsf{m}}(x) = \varphi(\sigma(x^i))$ *for* $x^i \in X_1$.

Lemma 20. *Let* $t \in \mathrm{T}(\mathcal{F}, \mathcal{V}^*)$ *such that* $\mathcal{V}(t) \cap X_0 = \emptyset$ *and* $\mathcal{V}^*(t) \setminus \mathcal{V} \subseteq X_2$. *Then* $[\![t]\!]^{\sigma_{\mathsf{u}}}_{\sigma_{\mathsf{m}}} = \varphi(t)$. *In particular, for any* $x^i \in X_1$, $[\![\sigma(x^i)]\!]^{\sigma_{\mathsf{u}}}_{\sigma_{\mathsf{m}}} = \varphi(\sigma(x^i))$.

The next two key lemmas are used to prove the theorem below.

Lemma 21. *For any* $t \in \mathrm{T}(\mathcal{F}, \mathcal{V}^*)$ *and any* ∇*-substitution* σ, $[\![\sigma^*(t)]\!]^{\sigma_{\mathsf{u}}}_{\sigma_{\mathsf{m}}} = [\![t]\!]^{\sigma_{\mathsf{u}}}_{\sigma_{\mathsf{m}}}$.

Proof. First prove $[\![\sigma(t)]\!]^{\sigma_{\mathsf{u}}}_{\sigma_{\mathsf{m}}} = [\![t]\!]^{\sigma_{\mathsf{u}}}_{\sigma_{\mathsf{m}}}$ using Lemmas 15, 19 and 20. ☐

Lemma 22. *Let* $s, t \in \mathrm{T}(\mathcal{F}, \mathcal{V}^*)$, *and* ρ, τ *be substitutions. Suppose that* $[\![l]\!]^{\tau}_{\rho} = [\![r]\!]^{\tau}_{\rho}$ *for any* $l \approx r \in E$. *Then if* $u \approx_E v$ *then* $[\![u]\!]^{\tau}_{\rho} = [\![v]\!]^{\tau}_{\rho}$.

Proof. By induction on the derivation of $u \approx_E v$, using Lemma 15. ☐

Theorem 23 (consistency and semi-unifiability). *For any terms* $s, t \in \mathrm{T}(\mathcal{F}, \mathcal{V})$, *the following are equivalent: (i)* $\{\nabla(s) \approx t\}$ *is semi-unifiable, (ii)* $\{s \leqslant t\}$ *is semi-unifiable, and (iii)* $\{\nabla(s) \approx t\}$ *is consistent.*

Proof. $(iii) \Rightarrow (i)$ will be shown later (Corollary 45). To show $(i) \Rightarrow (ii)$, use Lemmas 15 and 21. To show $(ii) \Rightarrow (iii)$, use Lemmas 15 and 22. ☐

$(ii) \Leftrightarrow (iii)$ was obtained in [8]; we incorporate an equivalence with (i).

5 Partial Correctness of Symbolic Semi-Unification

In this section, we give a rule-based symbolic semi-unification procedure and show its partial correctness. Our calculus is a variant of the one given in [8]. Essentially the same calculi are given in [12, 13, 15, 19]. Before giving the procedure, we need a preparation.

Definition 24 (relation $\gtrdot$). *We fix an arbitrary (strict) total order* $\gtrdot$ *on* $\mathcal{V}^*$ *satisfying (i)* $i > j$ *implies* $x^i \gtrdot x^j$ *and (ii)* $x^i \gtrdot y^j$ *implies* $x^{i+1} \gtrdot y^{j+1}$. *The order* $\gtrdot$ *is extended by* $x^i \gtrdot t$ *for any* $t \notin \mathcal{V}^*$ *and* $x^i \not\trianglelefteq t$.

$$\text{Decompose} \quad \frac{\{f(s_1,\ldots,s_n) \approx f(t_1,\ldots,t_n)\} \uplus E}{\{s_1 \approx t_1,\ldots,s_n \approx t_n\} \cup E}\ f \in \mathcal{F}$$

$$\text{Reduce} \quad \frac{\{x^i \approx t, C[x^i] \approx u\} \uplus E}{\{x^i \approx t, C[t] \approx u\} \cup E}\ x^i \gtrdot t$$

$$\text{Delete} \quad \frac{\{x^i \approx x^i\} \uplus E}{E}$$

$$\text{Clash} \quad \frac{\{f(s_1,\ldots,s_m) \approx g(t_1,\ldots,t_n)\} \uplus E}{\bot}\ f \neq g,\ f,g \in \mathcal{F}$$

$$\text{Check} \quad \frac{\{x^i \approx t\} \uplus E}{\bot}\ t \notin \mathcal{V}^*,\ x^i \trianglelefteq t$$

Fig. 1. Inference rules for symbolic semi-unification

It readily follows from our condition that (i) $i > j$ iff $x^i \gtrdot x^j$ and (ii) $x^i \gtrdot y^j$ iff $x^{i+1} \gtrdot y^{j+1}$. One way to give $\gtrdot$ is to fix some total order $>$ on $\mathcal{V}$ and define $x^i \gtrdot y^j$ iff either $x > y$ or ($x = y$ and $i > j$) as in [8, 15, 19]. But our proof reveals that the abstract condition above is sufficient.

Definition 25 (symbolic semi-unification procedure). *One step derivation using any of inference rules listed in Figure 1 is denoted by $\rightsquigarrow$. Here, the inference rules act on a finite set of ∇-equations and $\uplus$ denotes the disjoint union. For an input of a finite set E_0 of ∇-equations and the relation $\gtrdot$, a symbolic semi-unification procedure non-deterministically constructs a derivation $E_0 \rightsquigarrow E_1 \rightsquigarrow \cdots$ (possibly following some fixed derivation strategy). The derivation may be finite or infinite, and it is* maximal *if it does not end with E_k for which a further application of an inference rule is possible. A symbolic semi-unification procedure (following a fixed derivation strategy)* terminates *if any derivation (following that derivation strategy) is finite.*

The reflexive transitive closure of $\rightsquigarrow$ is denoted by $\overset{*}{\rightsquigarrow}$.

Example 26. Let the total order $\gtrdot$ be given by $w^i \gtrdot x^j \gtrdot y^k \gtrdot z^l$ for any i, j, k, l. Consider $E = \{y^3 \approx z, w^3 \approx x, x^2 \approx \mathsf{f}^2(y), x^1 \approx \mathsf{f}(w^2)\}$. Then we have

$$\begin{aligned}
&\{y^3 \approx z, w^3 \approx x, x^2 \approx \mathsf{f}^2(y), x^1 \approx \mathsf{f}(w^2)\} \\
&\quad \rightsquigarrow \{y^3 \approx z, w^3 \approx x, \underline{\mathsf{f}(w^3)} \approx \mathsf{f}^2(y), x^1 \approx \mathsf{f}(w^2)\} \\
&\quad \rightsquigarrow \{y^3 \approx z, w^3 \approx x, \underline{w^3 \approx \mathsf{f}(y)}, x^1 \approx \mathsf{f}(w^2)\} \\
&\quad \rightsquigarrow \{y^3 \approx z, \underline{\mathsf{f}(y)} \approx x, w^3 \approx \mathsf{f}(y), x^1 \approx \mathsf{f}(w^2)\} \\
&\quad \rightsquigarrow \{y^3 \approx z, \underline{x} \approx \mathsf{f}(y), w^3 \approx \mathsf{f}(y), \underline{\mathsf{f}(y^1)} \approx \mathsf{f}(w^2)\} \\
&\quad \rightsquigarrow \{y^3 \approx z, x \approx \mathsf{f}(y), w^3 \approx \mathsf{f}(y), \underline{w^2 \approx y^1}\} \\
&\quad \rightsquigarrow \{y^3 \approx z, x \approx \mathsf{f}(y), \underline{y^2} \approx \mathsf{f}(y), w^2 \approx y^1\} \\
&\quad \rightsquigarrow \{z \approx \underline{\mathsf{f}(y^1)}, x \approx \mathsf{f}(y), y^2 \approx \mathsf{f}(y), w^2 \approx y^1\}.
\end{aligned}$$

Here, modified parts are underlined.

Remark 27. In [8], the following (almost) rule-based procedure is given: (1) Apply Decompose as many times as possible; Apply Clash or Check if possible. (2) Fix a total order $>$ on variables. Consider a ground TRS $R = \{l \to r \mid l \approx r \in E, l > r\}$. Then: "For each rule, reduce each side (if reducible) by a single step of rewriting by other rules. If no rule can be rewritten any further, report SUCCESS. Otherwise replace the rule by the new equation thus obtained and go to Step (1)."

It is claimed in [8] that this procedure is terminating. The description "For each rule, ..." is difficult to interpret: for example,

$$\begin{aligned}\{\mathsf{h}(\mathsf{g}(y^1),\mathsf{g}(z^1),\mathsf{g}(x^1)) \approx \mathsf{h}(x,y,z)\} &\rightsquigarrow \{x \approx \underline{\mathsf{g}(y^1)}, y \approx \underline{\mathsf{g}(z^1)}, z \approx \underline{\mathsf{g}(x^1)}\}\\ &\rightsquigarrow \{x \approx \overline{\mathsf{g}^2(z^2)}, y \approx \overline{\mathsf{g}^2(x^2)}, z \approx \overline{\mathsf{g}^2(y^2)}\}\\ &\rightsquigarrow \{x \approx \underline{\mathsf{g}^4(y^4)}, y \approx \underline{\mathsf{g}^4(z^4)}, z \approx \underline{\mathsf{g}^4(x^4)}\}\\ &\rightsquigarrow \cdots\end{aligned}$$

should not be the case, as they claim that their procedure terminates.

Remark 28. Recall the following inference rules of the unification procedure corresponding to Reduce and Check:

$$\text{Reduce}' \quad \frac{\langle\{x \approx t\} \uplus E, \sigma\rangle}{\langle\{x := t\}(E), \{x := t\} \circ \sigma\rangle}\ x \notin \mathcal{V}(t) \qquad \text{Check}' \quad \frac{\langle\{x \approx s\} \uplus E, \sigma\rangle}{\bot}\ x \trianglelefteq s, s \notin \mathcal{V}$$

Two differences can be observed:

- the equations part and the substitution part are separated, and
- Reduce′ uses the substitution (replacing all occurrences of x), while Reduce uses the replacement (replacing a single occurrence of x^i).

In the unification procedure, the substitution part can be naturally separated from the equations part, as the substitution $\{x := t\}$ is not needed again in the equation part. But in the case of semi-unification, this is not the case:

$$\begin{array}{ll} \{x^1 \approx \mathsf{f}(x,y^2), y^1 \approx \mathsf{g}(x), y^3 \approx \mathsf{g}(z)\} & \{x^1,x,y^2,y^1,x,y^3,z\}\\ \rightsquigarrow \{x^1 \approx \mathsf{f}(x,y^2), y^1 \approx \mathsf{g}(x), \underline{\mathsf{g}(x^2)} \approx \mathsf{g}(z)\} & \{x^1,x,y^2,y^1,x,x^2,z\}\\ \rightsquigarrow \{x^1 \approx \mathsf{f}(x,y^2), y^1 \approx \mathsf{g}(x), \underline{x^2 \approx z}\} & \{x^1,x,y^2,y^1,x,x^2,z\}\\ \rightsquigarrow \{x^1 \approx \mathsf{f}(x,y^2), y^1 \approx \mathsf{g}(x), \underline{\mathsf{f}(x^1,y^3)} \approx z\} & \{x^1,x,y^2,y^1,x,x^1,y^3,z\}\\ \rightsquigarrow \cdots & \end{array}$$

At the first line, $x^1 := \mathsf{f}(x,y^2)$ can not be applied to other equations, while it can be applied to the equation $x^2 \approx z$ at the third line. Thus, a solved equation needs to be kept in the system for the future simplifications or for the case the equation itself is simplified in the future. Hence, it is not possible to split off the solved equations as the substitution part. Furthermore, since the substitution does not eliminate the future need of the application of the same substitution, it seems natural to use the replacement in Reduce instead of the substitution.

It is also observed that the derivation from the first line to the fourth line strictly increases the multiset of ∇-variables in the equation, which is listed at the right. Hence it is difficult to adapt a termination proof similar to the one applied to calculi for unification (see e.g. [1]).

Example 29. In fact, our calculus is not terminating—this is witnessed by the following derivation.

$$
\begin{array}{l}
\{x \approx z, x \approx \mathsf{g}(y), y \approx \mathsf{g}(z), z \approx \mathsf{g}(x)\} \\
\quad \rightsquigarrow \{\underline{\mathsf{g}(y)} \approx z, x \approx \mathsf{g}(y), y \approx \mathsf{g}(z), z \approx \mathsf{g}(x)\} \\
\quad \rightsquigarrow \{\mathsf{g}(y) \approx \underline{\mathsf{g}(x)}, x \approx \mathsf{g}(y), y \approx \mathsf{g}(z), z \approx \mathsf{g}(x)\} \\
\quad \rightsquigarrow \{\underline{y \approx x}, x \approx \mathsf{g}(y), y \approx \mathsf{g}(z), z \approx \mathsf{g}(x)\} \\
\quad \rightsquigarrow \{\underline{\mathsf{g}(z)} \approx x, x \approx \mathsf{g}(y), y \approx \mathsf{g}(z), z \approx \mathsf{g}(x)\} \\
\quad \rightsquigarrow \{\mathsf{g}(z) \approx \underline{\mathsf{g}(y)}, x \approx \mathsf{g}(y), y \approx \mathsf{g}(z), z \approx \mathsf{g}(x)\} \\
\quad \rightsquigarrow \{\underline{z \approx y}, x \approx \mathsf{g}(y), y \approx \mathsf{g}(z), z \approx \mathsf{g}(x)\} \\
\quad \rightsquigarrow \{\underline{\mathsf{g}(x)} \approx y, x \approx \mathsf{g}(y), y \approx \mathsf{g}(z), z \approx \mathsf{g}(x)\} \\
\quad \rightsquigarrow \{\mathsf{g}(x) \approx \underline{\mathsf{g}(z)}, x \approx \mathsf{g}(y), y \approx \mathsf{g}(z), z \approx \mathsf{g}(x)\} \\
\quad \rightsquigarrow \{\underline{x \approx z}, x \approx \mathsf{g}(y), y \approx \mathsf{g}(z), z \approx \mathsf{g}(x)\} \\
\quad \rightsquigarrow \cdots
\end{array}
$$

We note that because of some postponed applications of Reduce, Check is not applicable in the derivation. This infinite derivation is also valid in the calculus given in [15]. In Section 6, we will give a sufficient criterion on derivation strategies under which any derivation terminates.

Remark 30. The infinite derivation above is not possible, if one adopts a variant of Reduce using substitution (instead of the replacement):

$$
\text{Reduce}'' \quad \frac{\{x^i \approx t\} \uplus E}{\{x^i \approx t\} \cup \{x^i := t\}(E)} \; x^i \gtrdot t
$$

Rule-based semi-unification calculi in [8, 15] use the replacement, and those in [12, 13, 19] use the substitution. We note that any substitution can be simulated by repeated applications of replacement. We refer Lemma 43 for the termination of the calculus obtained by replacing the Reduce by Reduce$''$—termination of such calculus under a particular derivation strategy is also obtained in [12].

Remark 31. Another difference of the rule-based semi-unification calculi in the literature is whether the transformation $\nabla(f(t_1, \ldots, t_n)) = f(\nabla(t_1), \ldots, \nabla(t_n))$ is admitted in the course of derivations. The calculi in [15, 19] admit such flexibility. Adding such flexibility, however, causes another non-terminating derivation[2].

We now give several properties of finite derivations.

Lemma 32. *Suppose $E \overset{*}{\rightsquigarrow} E'$ with $E' \neq \bot$. Then $\approx_E = \approx_{E'}$.*

Using Lemma 13, we have

Corollary 33. *If $E \overset{*}{\rightsquigarrow} E' \neq \bot$, then E is semi-unifiable iff E' is semi-unifiable.*

Using Lemma 13 and Corollary 33, partial correctness of our symbolic semi-unification procedure is obtained.

Theorem 34 (partial correctness). *Let E be a finite set of ∇-equations. (1) If $E \overset{*}{\rightsquigarrow} \bot$ then E is not semi-unifiable. (2) If $E \overset{*}{\rightsquigarrow} E' \neq \bot$ and no inference rules are applicable to E', then E is semi-unifiable.*

[2] The authors learned this observation from an anonymous reviewer.

6 Termination of Symbolic Semi-Unification Procedure

In this section, we show termination of our rule-based symbolic semi-unification procedure under an assumption on the derivation strategy employed.

For the proof, we introduce a new relation.

Definition 35 (variable relation). *A* relation $>_v$ *on* ∇*-variables, called the* variable relation consistent with a set of ∇-equations E and the order $\gtrdot$, *is the smallest transitive relation satisfying the following conditions: (i) if* $x^i \approx_E y^j$ *with* $x^i \gtrdot y^j$ *then* $x^i >_v y^j$*, (ii) if* $x^i \approx_E C[y^j]$ *with* $C[y^j] \notin \mathcal{V}^*$ *for some strict context* C *then* $x^i >_v y^j$*. Here, a context* C *is* strict *if* C *is not of the form* $C'[\nabla(\Box)]$ *for some context* C'*. The reflexive closure of* $>_v$ *is denoted by* $\geq_v$*.*

Lemma 36. *Let* $>_v$ *be a variable relation consistent with* E *and* $\gtrdot$*. If* $x^i >_v y^j$ *then* $x^i \approx_E C[y^j]$ *for some context* C*; furthermore, if* $C = \Box$ *then* $x^i \gtrdot y^j$*.*

Lemma 37. *Let* X *be a finite set of variables. Let* $a_0, a_1, \ldots$ *be an infinite sequence of* ∇*-variables from* $X^* = \{x^i \mid x \in X, i \geq 0\}$*. Then there exist indexes* i, j *with* $i < j$ *such that* $a_j = \nabla^k(a_i)$ *for some* $k \geq 0$*.*

Definition 38 ($\mathcal{M}(t)$, $\mathcal{M}(E)$). *The multiset* $\mathcal{M}(t)$ *of* ∇*-variables that occur* maximally *in a* ∇*-term* t *is defined like this:* $\mathcal{M}(x^i) = \{x^i\}$*;* $\mathcal{M}(f(t_1, \ldots, t_n)) = \biguplus_i \mathcal{M}(t_i)$*. For a finite set* E *of* ∇*-equations, we put* $\mathcal{M}(E) = \biguplus\{\mathcal{M}(s) \uplus \mathcal{M}(t) \mid s \approx t \in E\}$*. Here,* $\uplus$ *denotes the multiset union.*

The next property is well-known (see e.g. [1]).

Proposition 39. *Let* $\gg_v$ *be the multiset extension of* $>_v$ *and* $\gtrsim_v$ *its reflexive closure. Let* $M_0 \gtrsim_v M_1 \gtrsim_v \cdots$ *be an infinite sequence of finite multisets such that* $M_i \gg_v M_{i+1}$ *for infinitely many indexes* i*. Then there is an infinite sequence* $a_0 \geq_v a_1 \geq_v \cdots$ *with* $a_i \in M_i$ *such that* $a_i >_v a_{i+1}$ *for infinitely many indexes* i*.*

Lemma 40. *Let* E_0 *be a finite set of* ∇*-equations,* $>_v$ *the variable relation consistent with* E_0 *and* $\gtrdot$*. If* $E_0 \overset{*}{\rightsquigarrow} E \rightsquigarrow E' \neq \bot$ *then* $\mathcal{M}(E) \gtrsim_v \mathcal{M}(E')$*. In particular, if* $E \rightsquigarrow E'$ *is by Reduce, then* $\mathcal{M}(E) \gg_v \mathcal{M}(E')$*.*

Proof. Distinguish the cases by the inference rule used in $E \rightsquigarrow E'$. □

Theorem 41 (termination of symbolic semi-unification procedure). *Every derivation starting from a consistent finite set of* ∇*-equations is finite.*

Proof. Suppose $E_0 \rightsquigarrow E_1 \rightsquigarrow \cdots$ be an infinite derivation and E_0 is consistent. Then, Clash and Check can not be used in this derivation. Decompose and Delete do not increase $\mathcal{M}(E_i)$ but reduce the number of symbols. Hence there does not exists an index j such that the Decompose and Delete are used for all $E_i \rightsquigarrow E_{i+1}$ with $i > j$. Thus there are infinitely many i such that Reduce is used on $E_i \rightsquigarrow E_{i+1}$. Hence $\mathcal{M}(E_0) \gtrsim_v \mathcal{M}(E_1) \gtrsim_v \cdots$ and there are infinitely many i such that $\mathcal{M}(E_i) \gg_v \mathcal{M}(E_{i+1})$ by Lemma 40. Then, by Proposition 39, we have an infinite sequence $x_0^{i_0} >_v x_1^{i_1} >_v \cdots$. Thus by Lemma 37 there exists indexes

k, l ($k < l$) such that $i_k \leq i_l$ and $x_k = x_l = x$. Since $>_v$ is transitive relation, $x_k^{i_k} >_v x_l^{i_l}$. Hence by Lemma 36, $x^{i_k} \approx_E C[x^{i_l}]$ for some context C such that $C = \Box$ implies $x^{i_k} \gtrdot x^{i_l}$. $C \neq \Box$ contradicts with the consistency of E. If $C = \Box$ then $x^{i_k} \gtrdot x^{i_l}$ contradicts $i_k \leq i_l$. Thus there exists no infinite derivation. $\Box$

The above theorem motivates the following definition.

Definition 42 (refutational completeness). *A derivation strategy is said to be* refutationally complete *if any maximal derivation starting from an inconsistent set of* ∇*-equations and following that strategy is finite and ends with* $\bot$*.*

The next lemma gives a concrete example of refutationally complete derivation strategy. Note that the rule Reduce$''$ is given in Remark 30.

Lemma 43. *A derivation strategy subject to using Reduce$''$ in place of Reduce and applying Check whenever possible is refutationally complete.*

Proof. Suppose $E_0 \rightsquigarrow E_1 \rightsquigarrow \cdots$ be a maximal derivation and E_0 is inconsistent. If Clash or Check is used then we are done. Otherwise, from the proof of Theorem 41, we have a sequence of ∇-variables $x^{i_k} = x_k^{i_k} >_v x_{k+1}^{i_{k+1}} >_v \cdots >_v x_l^{i_l} = x^{i_l}$ with $i_k \leq i_l$. This means that there exist $x^{i_k} \approx C_k[x_{k+1}^{i_{k+1}}] \in E_{i_k}, \ldots, x_{l-1}^{i_{l-1}} \approx C_l[x^{i_l}] \in E_{i_{l-1}}$. Since Reduce$''$ is simulated in our derivation, every $x_{j+1}^{i_{j+1}}$ in $C_j[x_{j+1}^{i_{j+1}}]$ is replaced by $C_{j+1}[x_{j+2}^{i_{j+2}}]$ for $j = k+1, \ldots, l-1$. Hence $x^{i_k} \approx C_k[x_{k+1}^{i_{k+1}}] \in E_{i_k}$ has the descendant $x^{i_k} \approx C[x^{i_l}] \in E_{i_l}$ such that $C = \Box$ implies $x^{i_k} \gtrdot x^{i_l}$. If $C \neq \Box$ then Check can be applied, which contradicts our assumption. If $C = \Box$ then $x^{i_k} \gtrdot x^{i_l}$ contradicts $i_k \leq i_l$. $\Box$

The next theorem immediately follows from Theorems 34 and 41.

Theorem 44 (total correctness). *The symbolic semi-unification procedure terminates if it follows a refutationally complete derivation strategy; either the input E is semi-unifiable and any maximal derivation ends with a set of ∇-equations or E is not semi-unifiable and any maximal derivation ends with $\bot$.*

We now obtain Theorem 4.1 of [8], on which their correctness proof is based, as a corollary.

Corollary 45 (consistency and semi-unifiability [8]). *Let E be a finite set of ∇-equations. Then E is consistent iff E is semi-unifiable.*

Proof. ($\Rightarrow$) Use Theorem 44 and Lemma 32. ($\Leftarrow$) Use Lemma 13. $\Box$

7 Conclusion

We have revisited rule-based calculi for uniform semi-unification, on which efficient uniform semi-unification procedures [8, 15] are based. We have given a new characterization of symbolic semi-unification and extended the correspondence between symbolic semi-unifiability and uniform semi-unifiability. For a rule-based calculus of symbolic semi-unification, which is given in a general form essentially including those of [8, 12, 15, 19], we have shown its termination and correctness under refutationally complete derivation strategy.

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